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Fluid-dynamical interpretation of quantum damped oscillators

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Abstract. The model of a quantum damped oscillator based on the Caldirola-Kanai equation is investigated by exploiting its coherent state basis. It is shown how these coherent states facilitate the derivation of fluid-dynamical equations corresponding to this model in the classical limit. From the conservative form of these equations it is concluded that the system represented by the Caldirola-Kanai equation cannot be considered as the quantum analogue of the classical damped oscillator. This is contrasted with the case of a harmonic oscillator interacting with a heat bath.

1. Introduction

There has been increasing interest in recent years in the problem of the quantum mechanical treatment of dissipative systems. The reason behind this is related to the fact that such a problem has applications in many widely separated fields such as nuclear physics, quantum optics and plasma physics. There is more than one method of dealing with this problem. One way is to consider the classical equation of motion of a damped harmonic oscillator

$$\ddot{x}+2\gamma_0\dot{x}+\omega_0^2x=0. \tag{1}$$

It is well known (Dekker 1981) that this equation can be derived from the Bateman Lagrangian

$$\mathscr{L}_0=\tfrac{1}{2}m_0\exp(2\gamma_0t)(\dot{x}^2-\omega_0^2x). \tag{2}$$

A Hamiltonian corresponding to $\mathscr{L}_0$ can also be constructed in the standard way, or

$$H_0=p^2\exp(-2\gamma_0t)/2m_0+\tfrac{1}{2}m_0\exp(2\gamma_0t)\omega_0^2x^2 \qquad (p=m_0\exp(2\gamma_0t)\dot{x}). \tag{3}$$

The quantisation of H_0 in the conventional manner leads to the time-dependent Schrödinger equation

$$\left(-\frac{\hbar^2}{2m_0}\exp(-2\gamma_0t)\frac{\partial^2}{\partial x^2}+\tfrac{1}{2}m_0\exp(2\gamma_0t)\omega_0^2x^2\right)\psi(x,t)=\mathrm{i}\hbar\frac{\partial\psi(x,t)}{\partial t} \tag{4}$$

which is also known as the Caldirola-Kanai equation (Caldirola 1941, 1983). Since H_0 leads to the classical equation of motion (1) representing a damped system, some authors (Caldirola 1941, 1983, Dodonov and Man'Ko 1978, 1979) have claimed that (4) describes a quantum damped system. However, there is now accumulating evidence in favour of a conservative system interpretation of (4) (Senitzky 1960, Greenberger 1979, Ray 1979, Cerveró and Villarroel 1984).

Another model of a damped quantum system which has also received much attention in recent years is that of a harmonic oscillator interacting with a heat bath (Glauber

1969, Louisell 1969). Since damping or dissipation arises as a result of neglecting some channels of energy leakage from the system to the outside world, it is expected that this model of quantum damping is more representative of the situation. This is well supported by the work of Ghosh *et al* (1977) who have invoked the coherent states concept to derive fluid-dynamical equations for this system in the classical limit, showing that these equations contain dissipative terms well known in classical fluid dynamics.

In connection with the last point, it should be emphasised that the use of the coherent state basis to derive fluid-dynamical equations for quantum systems in the classical limit is now well established (Ghosh *et al* 1977, Habeeb 1987). However, to our knowledge, no attempt has been made to clarify some of the interpretation questions connected with damping in the Caldirola–Kanai oscillator by using the coherent state and fluid dynamics concepts in analogy with what has been done for the oscillator interacting with a heat bath. By doing so, we hope in the present paper to approach the interpretation problem from a new perspective. This would also allow us to compare two models of quantum damping by analysing them both in the same framework. This framework is the classical fluid dynamical interpretation within which concepts such as damping and dissipation have well defined meanings.

To this end we will employ the coherent state basis, already known for the Caldirola–Kanai oscillator from the work of Dodonov and Man'Ko (1979), to derive the relevant fluid-dynamical equations in the classical limit in § 2. Then, in § 3 we briefly review a similar treatment already known (Ghosh *et al* 1977) for the quantum oscillator interacting with a heat bath for comparison purposes. Finally, § 4 closes with a discussion.

2. Fluid-dynamical interpretation of the Caldirola–Kanai equation

The coherent states $\psi_\alpha(x, t)$ for the system described by (4) represent a special case of those constructed by Dodonov and Man'Ko (1979). They satisfy the equations

$$\hat{A}(t)\psi_\alpha(x, t) = \alpha\psi_\alpha(x, t) \tag{5}$$

$$\hat{A}^{+}(t)\psi_\alpha(x, t) = \frac{\partial}{\partial\alpha}[\exp(\tfrac{1}{2}|\alpha|^2)\psi_\alpha(x, t)]\exp(-\tfrac{1}{2}|\alpha|^2) \tag{6}$$

where $\hat{A}(t)$ and $\hat{A}^{+}(t)$ are boson annihilation and creation operators respectively, with

$$\hat{A}(t) = (\mathrm{i}/\sqrt{2})(\varepsilon(t)\hat{p} - \dot{\varepsilon}(t)\,\mathrm{e}^{2\gamma_0 t}\hat{x}) \tag{7}$$

where $\varepsilon(t)$ is a complex function satisfying

$$\ddot{b} + 2\gamma_0\dot{b} + \omega_0^2 b = 0 \tag{8}$$

$$\exp(2\gamma_0 t)(\dot{\varepsilon}\varepsilon^* - \dot{\varepsilon}^*\varepsilon) = 2\mathrm{i} \tag{9}$$

and $b(t) = 2^{-1/2}\varepsilon(t)$. It also immediately follows from the work of Dodonov and Man'Ko (1979) that the average values of the coordinate and momentum operators in these states are

$$\langle\alpha|\hat{x}|\alpha\rangle = \left(\frac{\hbar}{2m_0}\right)^{1/2}(\alpha\varepsilon^* + \alpha^*\varepsilon) \tag{10}$$

$$\langle\alpha|\hat{p}|\alpha\rangle = \left(\frac{\hbar m_0}{2}\right)^{1/2}\exp(2\gamma_0 t)(\alpha\dot{\varepsilon}^* + \alpha^*\dot{\varepsilon}). \tag{11}$$

Also, in the P representation, the density matrix $\hat{\mathscr{P}}$ for the system (4) can be written as (Glauber 1969)

$$\hat{\mathscr{P}} = \int d^2\alpha\, P(\alpha; t)|\alpha\rangle\langle\alpha|. \tag{12}$$

Using (7) and its Hermitian adjoint, the Hamiltonian operator corresponding to (3) becomes

$$H = \tfrac{1}{4}\hbar \exp(2\gamma_0 t)[f_1^* \hat{A}^2 + f_1 \hat{A}^{\dagger 2} + f_2(1 + 2\hat{A}^\dagger \hat{A})] \tag{13}$$

where

$$f_1 = \dot{\varepsilon}^2 + \varepsilon^2\omega_0^2 \tag{14}$$

$$f_2 = \dot{\varepsilon}^*\dot{\varepsilon} + \varepsilon\varepsilon^*\omega_0^2. \tag{15}$$

Now, we notice from (5), (6) and (12) that the same commutation relations for $\hat{A}$ and $\hat{A}^\dagger$ with $\hat{\mathscr{P}}$ hold as for the ordinary oscillator, or (Ghosh *et al* 1977, Habeeb 1987)

$$\hat{A}\hat{\mathscr{P}} = \int d^2\alpha\, |\alpha\rangle\langle\alpha|\alpha P(\alpha; t) \tag{16}$$

$$\hat{A}^\dagger\hat{\mathscr{P}} = \int d^2\alpha\, |\alpha\rangle\langle\alpha|\left(\alpha^* - \frac{\partial}{\partial\alpha}\right)P(\alpha; t) \tag{17}$$

$$\hat{\mathscr{P}}\hat{A} = \int d^2\alpha\, |\alpha\rangle\langle\alpha|\left(\alpha - \frac{\partial}{\partial\alpha^*}\right)P(\alpha; t) \tag{18}$$

$$\hat{\mathscr{P}}\hat{A}^\dagger = \int d^2\alpha |\alpha\rangle\langle\alpha|\alpha^* P(\alpha; t). \tag{19}$$

Then the Heisenberg equation of motion for $\hat{\mathscr{P}}$

$$\frac{\partial\hat{\mathscr{P}}}{\partial t} = -\frac{i}{\hbar}[H, \hat{\mathscr{P}}] \tag{20}$$

can be translated into the P representation as

$$\frac{\partial P}{\partial t} = -\frac{i\exp(2\gamma_0 t)}{4}\left[f_1^*\left(2\alpha\frac{\partial}{\partial\alpha^*} - \frac{\partial^2}{\partial\alpha^{*2}}\right) + f_1\left(\frac{\partial^2}{\partial\alpha^2} - 2\alpha^*\frac{\partial}{\partial\alpha}\right) + 2f_2\left(\alpha^*\frac{\partial}{\partial\alpha^*} - \alpha\frac{\partial}{\partial\alpha}\right)\right]P. \tag{21}$$

Considering α and α^* as functions of $\bar{x} \equiv \langle\alpha|\hat{x}|\alpha\rangle$ and $\bar{p} \equiv \langle\alpha|\hat{p}|\alpha\rangle$ we obtain from (10) and (11)

$$\frac{\partial}{\partial\alpha} = \left(\frac{\hbar}{2m_0}\right)^{1/2}\left(\varepsilon^*\frac{\partial}{\partial\bar{x}} + m_0\dot{\varepsilon}^*\exp(2\gamma_0 t)\frac{\partial}{\partial\bar{p}}\right) \tag{22}$$

and

$$\frac{\partial}{\partial\alpha^*} = \left(\frac{\hbar}{2m_0}\right)^{1/2}\left(\varepsilon\frac{\partial}{\partial\bar{x}} + m_0\dot{\varepsilon}\exp(2\gamma_0 t)\frac{\partial}{\partial\bar{p}}\right). \tag{23}$$

The classical limit is approached, as usual (Ghosh *et al* 1977, Habeeb 1987) by letting $\hbar \to 0$. In this limit quantal correlations vanish and $\bar{x}$ and $\bar{p}$ become the classical position coordinate (x) and classical momentum (p) respectively. Also, $P(\alpha; t) \to f(x, p; t)$ giving the classical distribution function in phase space (Ghosh *et al* 1977, Habeeb 1987). Therefore, in this limit (21) becomes

$$\frac{\partial f(x, p; t)}{\partial t} = \left(m_0 \omega_0^2 \exp(2\gamma_0 t) x \frac{\partial}{\partial p} - \frac{\exp(-2\gamma_0 t)}{m_0} p \frac{\partial}{\partial x} \right) f(x, p; t) \tag{24}$$

where (14) and (15) have been used. Converting from momentum space to velocity space using $v \equiv \dot{x} = p \exp(-2\gamma_0 t)/m_0$ (see (3)), we obtain from (24)

$$\frac{\partial f(x, v; t)}{\partial t} = \left(\omega_0^2 x \frac{\partial}{\partial v} - v \frac{\partial}{\partial x} \right) f(x, v; t). \tag{25}$$

As in Ghosh *et al* (1977) and Habeeb (1987), to obtain the relevant fluid-dynamical equations in this classical limit we first define the mass density as

$$\rho(x; t) \equiv m_0 \int f(x, v; t)\, \mathrm{d}v \tag{26}$$

and the 'local hydrodynamic velocity' $v^{(h)}$ through

$$\rho v^{(h)}(x; t) \equiv m_0 \int f(x, v; t) v \, \mathrm{d}v. \tag{27}$$

Then, the zeroth moment of (25) over v-space gives

$$\frac{\partial \rho(x; t)}{\partial t} = -\frac{\partial}{\partial x} (\rho(x; t) v^{(h)} x; t) \tag{28}$$

which represents the equation of continuity. The first moment of (26) over v-space gives the momentum conservation equation

$$\frac{\partial}{\partial t} (\rho v^{(h)}) = -\omega_0^2 \rho x - \frac{\partial}{\partial x} (\rho v^{(h)2}). \tag{29}$$

Combining (28) and (29) we obtain (cf Ghosh *et al* 1977)

$$\frac{\partial v^{(h)}}{\partial t} + v^{(h)} \frac{\partial v^{(h)}}{\partial x} = -\omega_0^2 x \tag{30}$$

which is the Euler equation of classical fluid dynamics with the term on the right-hand side representing the driving force of an oscillator of frequency ω_0 on the fluid.

3. Fluid-dynamical interpretation of an oscillator interacting with a heat bath

This section briefly outlines the treatment of a one-dimensional version of a quantum damping model along the lines of Ghosh *et al* (1977). This is to facilitate the comparison with the results of § 2.

In its simplest version, an oscillator interacting with a reservoir or heat bath is governed by the Hamiltonian (Glauber 1969, Louisell 1969)

$$\hat{H} = \hbar\omega \hat{a}^\dagger \hat{a} + \sum_k \hbar\omega_k \hat{b}_k^\dagger \hat{b}_k + \hbar \sum_k (\lambda_k \hat{a}^\dagger \hat{b}_k + \lambda_k \hat{a} \hat{b}_k) \tag{31}$$

where ω is the oscillator frequency, $\hat{a}$ and $\hat{a}^\dagger$ are its annihilation and creation operators respectively, $\{\omega_k\}$ are the frequencies of the set of oscillators representing the reservoir,

$\{\hat{b}_k\}$ and $\{\hat{b}_k^\dagger\}$ are their annihilation and creation operators respectively and $\{\lambda_k\}$ are coupling constants. Using the Wigner-Weisskopf approximation (Glauber 1969), Ghosh *et al* (1977) have obtained the equation of motion for the density operator $\hat{\mathscr{P}}$ for this system in the interaction picture. From this equation they then proved that

$$\partial\hat{S}/\partial t = \tfrac{1}{2}\gamma[2\hat{a}\hat{S}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{S} - \hat{S}\hat{a}^\dagger\hat{a}] + \gamma\bar{n}[\hat{a}^\dagger\hat{S}\hat{a} + \hat{a}\hat{S}\hat{a}^\dagger - \hat{a}^\dagger\hat{a}\hat{S} - \hat{S}\hat{a}\hat{a}^\dagger] \tag{32}$$

where $\hat{S}$ is the reduced density operator for the oscillator (31), which is the trace of $\hat{\mathscr{P}}$ over the reservoir modes,

$$\gamma = 2\pi g(\omega)|\lambda_\omega|^2 \tag{33}$$

and

$$\bar{n} = 1/[\exp(\hbar\omega/kT) - 1] \tag{34}$$

is the occupancy of the reservoir mode with frequency ω, $g(\omega)$ is the density of modes, and T is the reservoir's temperature. In deriving (32) certain simplifying assumptions have been made and the reader is referred to Ghosh *et al* (1977) for more details. Next, expressing (32) in the coherent state basis, for the oscillator with frequency ω, reverting to the Schrödinger picture and following a procedure similar to that of § 2, Ghosh *et al* (1977) obtained

$$\frac{\partial}{\partial t} S_s(\alpha, \alpha^*; t) = -\mathrm{i}\omega\left(\alpha^*\frac{\partial}{\partial\alpha^*} - \alpha\frac{\partial}{\partial\alpha}\right)S_s + \gamma\left[1 + \tfrac{1}{2}\left(\alpha\frac{\partial}{\partial\alpha} + \alpha^*\frac{\partial}{\partial\alpha}\right) + \bar{n}\frac{\partial^2}{\partial\alpha\partial\alpha^*}\right]S_s. \tag{35}$$

Defining the 'fluid density' and the 'local hydrodynamic velocity' and going to the classical limit as in § 2, one obtains from the zeroth v-moment

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x}(\rho v^{(h)}) = \tfrac{1}{2}\gamma\frac{\partial}{\partial x}(\rho x) \tag{36}$$

and from the first v-moment

$$\frac{\partial}{\partial t}(\rho v^{(h)}) = -\omega^2\rho x - \frac{\partial}{\partial x}(\rho v^{(h)2}) + \frac{\gamma}{2}\frac{\partial}{\partial x}(\rho v^{(h)}x) - \tfrac{1}{2}\gamma\rho v^{(h)}. \tag{37}$$

The last two equations correspond respectively to (25) and (26) of Ghosh *et al* (1977) in the limit $\hbar \to 0$. From (36) and (37) we obtain

$$\frac{\partial v^{(h)}}{\partial t} = -\omega^2 x - v^{(h)}\frac{\partial v^{(h)}}{\partial x} + \tfrac{1}{2}\gamma\left(x\frac{\partial v^{(h)}}{\partial x}\right) - \tfrac{1}{2}\gamma v^{(h)}. \tag{38}$$

4. Discussion

Examination of (30) shows that the quantum oscillator described by the Caldirola-Kanai equations admits in the classical limit a fluid-dynamical interpretation in terms of an Euler equation with no damping terms. This is in contrast to the case of an oscillator interacting with a heat bath whose fluid dynamics in the classical limit is governed by (38) in which the existence of damping terms is easily recognised. This supports the already existing arguments (Senitzky 1960, Ray 1979, Cerveró and Villarroel 1984) that the Caldirola-Kanai model cannot be considered as the quantum analogue of the classical damped oscillator. The advantage of the present treatment is that two models of damping can both be compared in the same scheme. It is hoped that such a treatment can be extended to other models of damping, such as the non-linear Schrödinger equation approach. However, such an extension is not expected to be straightforward because of the non-availability of the relevant coherent states. Also, it is of interest to apply the present idea to the wider class of harmonic oscillators with variable damping governed by a generalised Caldirola–Kanai equation for which coherent states are already available (Dodonov and Man'Ko 1979). Finally, it should be stated that the present work represents one example of the application of the coherent states of the Caldirola–Kanai oscillator which may motivate other applications elsewhere.

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